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CLASSICAL BUMP CEPHEIDS: RECONCILIATION OF THEORY WITH OBSERVATIONS

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ABSTRACT

New models of classical bump Cepheids lying along the Cepheid ridge line in the H-R diagram have been computed to obtain a more accurate set of predicted surface velocity curves. Statistical properties of these velocity curves and of the models' fundamental astrophysical parameters are compared with published Cepheid observations of the highest available quality. Assuming the validity of the adopted opacities for the main pulsating layers, acceptable Cepheid models are found to be 0.5 mag brighter at a fixed stellar mass than standard evolutionary models. By accepting the implied luminosity increase, all the previously obtained manifestations of the Cepheid mass discrepancy—with the exception of the anomalous double-mode (beat) Cepheid masses—are resolved, at least in a statistical sense. Newly published orbital masses of individual Cepheids in binary systems are similarly compatible with the new pulsational masses.

The need for a luminosity increase of 0.5 mag has been independently inferred for standard evolutionary models of intermediate-mass and high-mass giant and supergiant stars from their locations in the observational H-R diagram. A 0.5 mag increase is also fully consistent with the most recent evolutionary tracks including convective core overshooting, although it can also be achieved by changing the initial chemical composition. This degree of interagreement implies that the traditionally accepted zero point of observed Cepheid luminosities is correct.

Subject headings: opacities — stars: Cepheids — stars: interiors — stars: luminosities — stars: pulsation

I. INTRODUCTION

Theoretical models of classical bump Cepheids are known to contain a systematic error of unknown source. Past studies have shown that if the masses used to compute the models are adopted from a standard mass-luminosity relation for evolved giants, the observed sequence of light-curve shapes as a function of period (Hertzsprung 1926) can be reproduced only by models having periods that are significantly longer than those actually observed. The discrepancy for models built with Los Alamos opacities is ~140% (Stobie 1969*a*, *b*) while with the newer Carson opacities it is ~40% (Carson and Stothers 1984). Converted into a mass discrepancy, it amounts to masses that are too small by ~50% (Christy 1966*a*, 1968, 1975; Stobie 1969*a*, *b*; Fricke, Stobie, and Strittmatter 1971, 1972; Karp 1975*a*; Davis and Davison 1978; Simon and Davis 1983) and by ~15% (Vemury and Stothers 1978; Carson and Stothers 1984) in the two cases, respectively.

Since a mass discrepancy of 15% lies within the estimated range of uncertainty of the evolutionary masses, exotic remedies like mass loss, magnetic fields, and surface helium enrichment (e.g., Cox 1980) no longer seem necessary in the case of the Carson opacities. Although these opacities are now known to contain errors for the high-temperature metal contribution (Carson *et al.* 1984), the opacity errors are not important for Cepheid pulsation calculations because the main pulsating layers are too cool to be affected by them. This point has been explicitly demonstrated for type II Cepheid models in a previous paper (Carson, Stothers, and Vemury 1981, § IV), but we have verified it here for classical Cepheid models by recomputing a 7.9 day model with an opacity table in which the spurious opacity "bump" arising from the high-temperature metal contribution was artificially excised. Consequently, the chief differences between the Carson and Los Alamos opacities that are relevant here concern the hydrogen and helium contributions. In the Carson opacities, the opacity maximum at the peak of hydrogen ionization ($T \approx 1 \times 10^4$ K) is somewhat reduced, while the pulsationally critical opacity bump in the region of second helium ionization ($T \approx 4 \times 10^4$ K) is significantly more pronounced than in the Los Alamos opacities. Owing to the success of these newer hydrogen and helium opacities in explaining other kinds of Cepheid-like variables, such as RR Lyr stars, BL Her stars, and W Vir stars, we believe that they can be safely used in classical Cepheid models to infer the existence of any remaining discrepancies that are not opacity-related.

In the present paper, we specify masses for classical bump Cepheids that are ~15% smaller than standard evolutionary masses or, equivalently, luminosities that are ~60% brighter than standard evolutionary luminosities. A new sequence of nonlinear models pulsating at full amplitude is calculated, in a fresh effort to reproduce the known basic properties of these stars, in an essentially statistical sense. Since there does not exist such an object as an "average" Cepheid, we will consider the closest approximation to it, namely, the set of ridge-line Cepheids, although these may not be representative of the average in every respect. Ridge-line Cepheids are those Cepheids that lie nearly along the center line of the instability strip in the H-R diagram.

Our selected approach here is to use only velocity curves of the models. There are several reasons for this choice. First, velocity variations are, in general, more reliably calculated than luminosity variations, which have a larger sensitivity to the rather coarse mass zoning and crude treatment of radiative transfer near the stellar surface. Second, both computed and observed velocity curves at any given period exhibit a smaller range of shapes than do the corresponding light curves. Third, the observed velocity curves form a progression with period that is analogous to the Hertzsprung progression of light curves. Although this progression of the velocity curves has been recognized in various ways for many years (Joy 1937; Payne-Gaposchkin 1951, 1954; Parenago 1954; Ledoux and Walraven 1958; Fricke, Stobie, and Strittmatter 1972), it is only recently that large numbers of high-resolution observations of radial velocities have become available for detailed analysis.

In §§ II–III of the present paper, the necessary mean relations between basic theoretical and observational quantities are assembled. In § IV, the new models are presented, and in § V they are compared with observations. An interpretation of the new models in terms of recently proposed revisions to the theory of stellar evolution is given in § VI, while § VII concludes the paper by consolidating our main results.

II. THEORETICAL RELATIONS AMONG BUMP CEPHEID PARAMETERS

a) *Evolution Theory*

The mass-luminosity relation is taken to be a linear formula fitted to theoretical models of post–main-sequence stars crossing the instability strip on the H-R diagram for the second time:

$$\log (L/L_{\odot}) = 0.66 + 3.50 \log (M/M_{\odot}) + \gamma , \qquad (1)$$

where γ is an adjustable constant. If $\gamma = 0$, our standard mass-luminosity relation (Carson and Stothers 1984) is obtained; it closely resembles the fits derived independently by King *et al.* (1973) and by Becker, Iben, and Tuggle (1977), which are $\log (L/L_{\odot}) = 0.68 + 3.48 \log (M/M_{\odot})$ and $\log (L/L_{\odot}) = 0.46 + 3.68 \log (M/M_{\odot})$, respectively.

Our previous relation was derived from stellar models having initial abundances of hydrogen, helium, and metals equal to $(X, Y, Z) = (0.739, 0.240, 0.021)$. Conventional physical assumptions were adopted for the models—in particular, no rotation, no magnetic fields, no mass loss, and no convective core overshooting. These various assumptions will affect the value of γ that is required, although the choice of opacities will not (Carson and Stothers 1976; Becker 1985).

b) *Atmosphere Theory*

The Stefan-Boltzmann law defining the effective temperature T_e of the photosphere is

$$L/L_{\odot} = (R/R_{\odot})^2 (T_e/T_{e\odot})^4 . \qquad (2)$$

Bolometric correction of the visual absolute magnitude is defined to be

$$\mathrm{BC} = M_{\mathrm{bol}} - M_V = 4.75 - 2.5 \log (L/L_{\odot}) - M_V . \qquad (3)$$

Since bump Cepheids show $T_e = 5400$–5800 K (Pel 1985), the bolometric correction is only 0.00 to -0.05 (Flower 1977; Buser and Kurucz 1978; Bell and Gustafsson 1978). We set BC = 0.00.

c) *Pulsation Theory*

The fundamental period of pulsation P can be accurately represented by a fitted formula,

$$P = \alpha (R/R_{\odot})^{7/4} (M/M_{\odot})^{-3/4} \text{ days} . \qquad (4)$$

The coefficient α is relatively insensitive to chemical composition and also to opacity, as $\alpha \approx 0.0235$ for the Cox-Stewart opacities (Christy 1966*b*; Cogan 1978) and $\alpha \approx 0.025$ for the Carson opacities (Vemury and Stothers 1978). Over the range of periods occupied by bump Cepheids the exponents of M and R can also be taken as fixed (Cogan 1970; Fricke, Stobie, and Strittmatter 1971; Cox, King, and Stellingwerf 1972).

The secondary bump on the surface velocity curve has considerable diagnostic potential if a phase parameter ϕ_v is defined as the fraction of a cycle, after zero velocity at minimum radius, of the *second* (but not necessarily the secondary) bump, plus unity. (The sign of the velocity vector is taken as positive outward from the stellar center and thus is opposite in sense to the observers' convention.) Then, for $\phi_v > 1.3$,

$$P\phi_v = \beta (R/R_{\odot}) \text{ days} . \qquad (5)$$

Using the Cox-Stewart opacities, $\beta \approx 0.245$ (Christy 1970, 1975; Fricke, Stobie, and Strittmatter 1972) or, using the Carson opacities, $\beta \approx 0.215$ (Vemury and Stothers 1978). β has a negligible dependence on metals abundance and only a weak dependence on helium abundance, since $\phi_v \propto Y^{0.2} Z^{0.0}$ (Fricke, Stobie, and Strittmatter 1971; Vemury and Stothers 1978). Christy (1975) and Whitney (1956) have given a physical interpretation of equation (5), although other interpretations are possible.

III. OBSERVATIONAL RELATIONS AMONG BUMP CEPHEID PARAMETERS

a) *Period-Luminosity Relation*

Theory suggests that an observational relation ought to exist between period, luminosity, and effective temperature (or, equivalently, color) (Sandage 1958). Owing to the small intrinsic color width of the instability strip and the relatively poorly determined interstellar-reddening corrections, the empirical color term is still somewhat in dispute. Therefore, it is safer at present to use an observational mean period-luminosity relation. We adopt

$$M_V = -1.34 - 2.85 \log P , \qquad (6)$$

where P is in days and the zero point at $M_V(0.8)$ has been taken to be the straight average of 26 different determinations (based on five independent observational methods) for Galactic Cepheids, as indicated in Table 1, which is an updated version of Stothers's (1983) Table 1; the new zero point agrees exactly with the older one and is very close to the traditionally accepted value, with an estimated error of only ± 0.10 mag. The average *slope* of the period-luminosity relation, based on 20 different (but all photoelectrically calibrated) determinations listed in Table 2, is found to be precisely the same for Large and Small Magellanic Cloud Cepheids as for Galactic Cepheids, despite the differences of chemical composition among the three galaxies. Sandage and Tammann (1968), Butler (1978), and Caldwell and Coulson (1986) have reached a similar conclusion about the possible equality of the three slopes.

b) *Period-Bump Phase Relation*

For 38 Galactic Cepheids, sufficiently accurate radial-velocity curves have been published that fine structure on the curves can be resolved. These variables are listed in Table 3 together with their periods, true velocity amplitudes (radial-velocity amplitudes multiplied by 24/17), bump phase parameters ϕ_v, and branch locations of the secondary bump (A = ascending branch, D = descending branch, X = no bump). Since the observed radial velocities need corrections for

TABLE 1

OBSERVED ZERO POINT OF THE PERIOD-LUMINOSITY RELATION FOR CLASSICAL CEPHEIDS

Method	Weight	M_V at $\log P = 0.8$	N	References
Four methods, 15 determinations	15	−3.62	...	Stothers 1983 (Table 1)
Galactic clusters	1	−3.92	27	Fernie and McGonegal 1983
	1	−3.73	15	Caldwell 1983
	1	−3.37	8	Balona and Shobbrook 1984
	1	−3.50	9	Schmidt 1984
	1	−3.52	29	Cester and Marsi 1984
	1	−3.46	23	Caldwell and Coulson 1987
Hβ photometry	1	−3.92	...	Eggen 1985
Modified Baade-Wesselink	1	−3.52	17	Ivanov 1981
	1	−3.60	...	Ivanov and Nikolov 1983
	1	−3.59	28	Gieren 1986*a*
Statistical parallax	1	−3.73	37	Karimova and Pavlovskaya 1981
All	...	−3.62	...	Weighted mean

NOTE.—N is the number of stars used.

geometrical projection and for limb darkening to convert them to true astrocentric velocities, we have adopted the standard combined correction factor of 24/17 (Getting 1934).

To have a consistent comparison with the theoretical models, bump measurements have been made on the *inverse* of the observed radial-velocity curve; this standard procedure follows Fricke, Stobie, and Strittmatter (1972). Our new list includes 16 bump Cepheids, of which 12 stars have $\phi_v > 1.3$; these obey the following mean relation:

$$\phi_v = 2.75P^{-0.29} \tag{7}$$

with an average deviation per star of only ± 0.03. Since the exponent of P is only weakly constrained by the data, we have simply assigned a value of -0.29 in order to have compatibility with the combined exponents of equations (1)–(6). Note that, in general, low-amplitude Cepheids do not exhibit secondary bumps on their velocity curves, just as they do not exhibit them on their light curves (Fernie and Chan 1986). This agrees with the results for our theoretical models of low-amplitude Cepheids (Vemury and Stothers 1978; Carson and Stothers 1984).

c) *Period-Radius Relations*

Ideally, the period-bump phase relation should include a dependence on radius, if equation (5) is a proper guide. Although empirical radii are rather uncertain, radii based on the Baade-Wesselink method and its variants are available for 11 of the 12 bump Cepheids with $\phi_v > 1.3$ (Fernie 1984; Gieren 1985*b*; Coulson, Caldwell, and Gieren 1986). Assuming the functional form of equation (5), we find for these 11 stars the average relation

$$P\phi_v = (0.203 \pm 0.015)(R/R_\odot) \text{ days} . \tag{8}$$

Theory also predicts a correlation between period and radius alone (Evans 1976*b*). Six observational determinations of the period-radius relation are listed in Table 4 from sources that have used luminosities and effective temperatures of Galactic cluster members to derive radii. The average relation

TABLE 2

OBSERVED SLOPE OF THE PERIOD-LUMINOSITY RELATION FOR CLASSICAL CEPHEIDS IN THREE GALAXIES

Method	$dM_V/d \log P$	N	References
Galactic clusters	−2.78	13	Sandage and Tammann 1969
	−2.90	14	van den Bergh 1977
	−3.01 ± 0.10	26	Stothers 1983
	−2.88 ± 0.08	27	Fernie and McGonegal 1983
	−2.89 ± 0.13	29	Cester and Marsi 1984
	−2.96 ± 0.11	30	Caldwell and Coulson 1986
Hβ photometry	−2.61	...	Eggen 1985
Modified Baade-Wesselink	−3.0	143	Opolski 1968
	−2.9	17	Ivanov 1981
	−3.0	...	Ivanov and Nikolov 1983
	−2.56 ± 0.19	28	Gieren 1986*a*
Large Magellanic Cloud	−3.07	18	Gascoigne 1969
	−2.90	17	Gascoigne and Shobbrook 1978
	−2.70	60	Butler 1978
	−2.79 ± 0.20	26	Martin *et al.* 1979
	−2.79	75	Wayman *et al.* 1984
	−2.91 ± 0.15	26	Caldwell and Coulson 1986
Small Magellanic Cloud	−2.88	18	Gascoigne 1969
	−2.66	62	Butler 1976
	−2.97 ± 0.16	44	Caldwell and Coulson 1986
All	−2.85	...	Unweighted mean

NOTE.—Individual reddenings, when available, were adopted in preference to statistical reddenings. N is the number of stars used.

TABLE 3

OBSERVED VELOCITY-CURVE PROPERTIES OF CLASSICAL CEPHEIDS

Variable	P (days)	ΔV^a (km s^{-1})	ϕ_v	Bump	Ref.
SU Cas	1.95	31	...	X	1, 2
EU Tau	2.10	25	...	X	3
DT Cyg	2.50	20	...	X	4, 5
RT Mus	3.09	47	...	X	6
SZ Tau	3.15	32	...	X	7
R TrA	3.39	38	...	X	8, 9
UX Car	3.68	45	...	X	6
RT Aur	3.73	49	...	X	7, 10
AD Gem	3.79	52	...	X	11
SU Cyg	3.85	51	...	X	12
α UMi	3.97	8	...	X[b]	13
AH Vel	4.23	25	...	X	14
XY Cas	4.50	47	...	X	15
V482 Sco	4.53	49	...	X	8, 9
S Cru	4.69	42	...	X	8, 9
δ Cep	5.37	55	1.70	D	16, 17, 18, 19
V Car	6.70	42	1.60:	D	20
η Aql	7.18	62	1.60	D	17, 21, 22, 23, 24
HD 14662	7.57	8	...	X[b]	25
W Sgr	7.59	56	1.58:	D	17, 20, 26, 27, 28
U Vul	7.99	44	1.55:	D	5
HD 9250	8.38	14	...	X[b]	25
S Sge	8.38	51	1.49	D	29
V500 Sco	9.32	51	1.44	D	28
S Nor	9.75	42	1.43	D	28, 30, 31
AQ Car	9.77	42	1.41:	D	32
β Dor	9.84	51	1.43	D	28, 33, 34
ζ Gem	10.15	42	1.38	A	17, 35, 36
XX Cen	10.95	47	1.37:	A	32
XY Car	12.44	58	1.21	A	32
TT Aql	13.75	69	1.13	A	32
X Cyg	16.39	80	0.94	A	16, 37, 38
XZ Car	16.65	78	0.92	A	32
Y Oph	17.12	24	...	X	39, 40
YZ Car	18.17	40	...	X	41
T Mon	27.02	71	...	X	42, 43, 44
l Car	35.56	54	...	X	27, 28, 45, 46
SV Vul	45.03	70	...	X	16, 43

[a] ΔV is the radial-velocity amplitude multiplied by 24/17.
[b] Low-amplitude variable.

REFERENCES.—(1) Abt 1959. (2) Gieren 1976. (3) Burki 1985. (4) Grasberger and Herbig 1952. (5) Sanford 1951. (6) Stobie and Balona 1979. (7) Gieren 1985*a*. (8) Gieren 1981. (9) Gieren 1982*a*. (10) Petrie 1934. (11) Imbert 1983. (12) Imbert 1984. (13) Roemer 1965. (14) Gieren 1977. (15) Imbert 1981. (16) Benz and Mayor 1982. (17) Jacobsen 1926. (18) Moore 1913. (19) Shane 1958. (20) Lloyd Evans 1982. (21) Evans 1976*a*. (22) Jacobsen and Wallerstein 1981. (23) Schwarzschild, Schwarzschild, and Adams 1948. (24) Wright 1899. (25) Burki and Benz 1982. (26) Jacobsen 1974. (27) Lloyd Evans 1968. (28) Stibbs 1955. (29) Herbig and Moore 1952. (30) Breger 1970. (31) Feast 1967. (32) Coulson, Caldwell, and Gieren 1985*a*. (33) Applegate 1927. (34) Gratton 1953. (35) Campbell 1901. (36) Jacobsen and Wallerstein 1982. (37) Abt 1978. (38) Duncan 1921. (39) Abt 1954. (40) Evans and Lyons 1986. (41) Coulson 1983*b*. (42) Coulson 1983*a*. (43) Sanford 1956. (44) Wallerstein 1972. (45) Dawe 1969. (46) Jacobsen 1934.

is $\log(R/R_\odot) = 1.130 + 0.701 \log P$, for P in days. Seventeen sources that have used the Baade-Wesselink method or its variants to derive radii provide the period-radius relations given in Table 5; these average to $\log(R/R_\odot) = 1.171 + 0.672 \log P$. Since the two methods thus yield very similar results, we adopt an unweighted average of all 23 determinations:

$$\log(R/R_\odot) = 1.160 + 0.680 \log P \,. \qquad (9)$$

Using considerably fewer data, Parsons and Bouw (1971) obtained a similar average result, $\log(R/R_\odot) = 1.173 + 0.652 \log P$. Although Evans (1976*b*) and Fernie (1984) regarded the differences between the results of their applications of the two methods as being significant enough to worry about, we do not, owing to our expectation of large statistical fluctuations among the various determinations and to our confidence in the strong law of large numbers for the two averages.

If an attempt, however, is made to use equations (8) and (9) in combination with equations (1)–(7), the bump Cepheid parameters become overdetermined. We will therefore use the two period-radius relations only for consistency checks.

TABLE 4

OBSERVED PERIOD-RADIUS RELATION FOR CLASSICAL CEPHEIDS BASED ON GALACTIC CLUSTER MEMBERSHIP

$d \log(R/R_\odot)/d \log P$	$\log(R/R_\odot)$ at $\log P = 0$	N	References
0.692 ± 0.034	1.131 ± 0.030	13	Woolley and Carter 1973
0.680 ± 0.028	1.138 ± 0.026	15	Evans 1976*b*, 1977
0.662	1.150	13	Balona 1977
0.70:	1.17:	17	Cogan 1978
0.649	1.150	13	Gieren 1982*b*
0.824 ± 0.010	1.042 ± 0.015	27	Fernie 1984
0.701	1.130	...	Unweighted mean

NOTE.—N is the number of stars used.

IV. NEW BUMP CEPHEID MODELS

For the present models of bump Cepheids, we have adopted a series of nine stellar masses, listed in Table 6. To each mass we have assigned a luminosity from equation (1). Combination of equations (1)–(7) gives for the zero-point correction to this luminosity

$$\gamma = -1.82 - 4.67 \log \alpha + 8.17 \log \beta \,. \qquad (10)$$

The same equations also yield an associated effective temperature,

$$\log T_e = 4.372 + 0.286 \log \alpha - 0.214 \log(M/M_\odot) \,. \qquad (11)$$

We adopt $\alpha = 0.025$ and $\beta = 0.215$ from the models of Vemury

TABLE 5

OBSERVED PERIOD-RADIUS RELATION FOR CLASSICAL CEPHEIDS BASED ON THE MODIFIED BAADE-WESSELINK METHOD

$d \log(R/R_\odot)/d \log P$	$\log(R/R_\odot)$ at $\log P = 0$	N	References
0.646	1.190	53	Kurochkin 1966
0.804 ± 0.026	1.083 ± 0.029	83	Latyshev 1966
0.560 ± 0.008	1.258 ± 0.009	10	Fernie 1968
0.705 ± 0.071	1.181 ± 0.068	15	Woolley and Carter 1973
0.646 ± 0.040	1.214 ± 0.043	16	Thompson 1975
0.555 ± 0.096	1.291 ± 0.094	14	Evans 1976*b*, 1977
0.602	1.213	41	Balona 1977
0.71 ± 0.07	1.11 ± 0.04	17	Ivanov 1981
0.696 ± 0.040	1.120 ± 0.034	10	Caccin *et al.* 1981
0.700 ± 0.020	1.167 ± 0.024	30	Sollazzo *et al.* 1981
0.695	1.139	15	Gieren 1982*b*
0.587 ± 0.022	1.244 ± 0.023	55	Fernie 1984
0.701	1.175	34	Burki 1985
0.786 ± 0.047	1.040 ± 0.039	20	Gieren 1985*b*
0.637 ± 0.021	1.215 ± 0.028	61	Coulson *et al.* 1985*b*
0.625 ± 0.063	1.209 ± 0.054	20	Coulson *et al.* 1986
0.77 ± 0.04	1.06 ± 0.03	28	Gieren 1986*b*
0.672	1.171	...	Unweighted mean

NOTE.—N is the number of stars used.

TABLE 6

FULL-AMPLITUDE PROPERTIES OF THE CEPHEID MODELS

$M/M_\odot$	$\log(L/L_\odot)$	$\log T_e$	$R/R_\odot$	P (days)	K.E. (10^{42} ergs)	$\Delta R/R$	V_{out} (km s^{-1})	V_{in} (km s^{-1})	ΔV (km s^{-1})	ϕ_v	Bump	P_2/P_0
4.7......	3.23	3.768	41	4.79	4.0	0.18	44	−49	93	1.71	D	0.556
5.0......	3.33	3.763	47	5.96	4.7	0.19	42	−54	96	1.64	D	0.547
5.5......	3.47	3.755	57	7.90	5.1	0.19	36	−47	83	1.54	D	0.533
5.9......	3.59	3.750	67	10.00	4.1	0.19	31	−37	68	1.48	D	0.520
6.1......	3.64	3.748	71	10.77	2.0	0.14	24	−28	52	1.45	D	0.514
6.3......	3.68	3.745	76	11.68	2.5	0.14	26	−26	52	1.34	A	0.510
6.6......	3.75	3.742	83	13.94	9.7	0.28	43	−47	90	1.15	A	0.502
7.0......	3.83	3.737	93	16.53	15.4	0.35	50	−45	95	0.92	A	0.493
7.5......	3.94	3.732	109	20.87	25.8	0.37	51	−53	104	0.92	A[a]	0.479

[a] Bump occurs only when $\Delta V > 80$ km s^{-1} in the rise to limiting amplitude.

and Stothers (1978), and therefore obtain $\gamma = 0.22$. (Observe that, in principle, only one bump Cepheid model is needed to determine α and β, and hence γ.) Additional assumptions are as in Vemury and Stothers (1978)—note that the needed opacity tables have been published by Carson and Stothers (1984). Only the fundamental mode of pulsation has been calculated in our present models, just as in our previous ones; for the ratio of the second-overtone and fundamental periods, P_2/P_0, we have used linear nonadiabatic theory.

Properties of the models pulsating at full amplitude are listed in Table 6. Surface velocity curves for all the models (shown in Fig. 1) reveal that the Hertzsprung bump first becomes prominent at a period of $P \approx 5$ days, crosses velocity maximum at $P \approx 11$ days, and becomes insignificant again for $P > 17$ days. In the transitional period range, $P = 10$–12 days, the velocity amplitudes are about half as large as at other periods. These results, it must be remembered, refer only to ridge-line Cepheids.

V. COMPARISON BETWEEN THEORY AND OBSERVATIONS

The theoretical velocity curves resemble closely the observed velocity curves for large-amplitude Cepheids, including the very oddly shaped curves at the transition periods of 10–12 days. To quantify the gross features of the theoretical curves, we have decomposed all the curves by fitting Fourier series of the form

$$V = A_0 - \sum_{i=1}^{N} A_i \sin\left[i\omega(t - t_0) + \phi_i\right] . \qquad (12)$$

Various orders of fit were tried and a value of $N = 5$ was finally selected. Following Simon and Teays (1983), we shall define two quantities that are combinations of the lowest-order Fourier coefficients:

$$\phi_{21} = \phi_2 - 2\phi_1 + 2\pi n , \quad R_{21} = A_2/A_1 , \qquad (13)$$

where n is an integer. These quantities are plotted as a function of period in Figures 2 and 3 for all nine models. Also shown are Simon and Teays's (1983) equivalent quantities for 11 observed Cepheids. Theory and observation agree well for ϕ_{21}. Although the theoretical values of R_{21} are exaggerated for periods of 6–11 days, the undulating sequence of theoretical values through the rest of the period range reproduces the general character of the observations fairly well. Considering the generally larger amplitudes of the theoretical models, which may affect the skewness of the velocity curves and hence their Fourier coefficients (Stellingwerf and Donohoe 1986), we judge the agreement with observations to be reasonably good.

Smaller details on the velocity curves provide still another test. The bump phase ϕ_v is a physically significant detail and is plotted in Figure 4 for the models of Table 6 and the observed stars of Table 3. There is almost perfect agreement everywhere, except at the transition periods of 10–12 days where the theoretical periods for fixed ϕ_v are too long by about 10%. A reduction of the adopted masses there by 3% could probably eliminate the discrepancy.

On the other hand, theoretical and observed velocity amplitudes for periods of 10–12 days are in excellent agreement, although outside this region the models show discrepantly large amplitudes. A possible answer to this problem probably does not lie with the multiplicative factor that converts the observed radial velocities to true astrocentric velocities, since this projection factor has been rather consistently determined to be essentially a constant, given variously as 1.41 (Getting 1934), 1.31 (Parsons 1972), 1.28 (Frolov 1974), 1.31 (Karp 1975*b*), and ~1.36 (Hindsley and Bell 1986) for Cepheids of all periods. It is also unlikely that our plotting of the theoretical velocity curves for a fixed mass layer is the culprit, since plotting the velocity curves for a fixed optical depth reveals no significant difference, as Karp (1975*a*) likewise found for his model of a 12 day Cepheid. Although convection has been ignored in our models, inclusion of it might be able to reduce somewhat the velocity amplitudes (Deupree 1980). But it would probably not do so in the required *selective* way.

A possible resolution of the difficulty is to assume that ridge-line Cepheids actually possess the largest amplitudes of all. Figure 5 displays a period-amplitude diagram for a large number of observed classical Cepheids whose radial velocity amplitudes were compiled from the literature by Cogan (1980). The observed amplitudes are plotted here after being multiplied by 24/17. Notice that the amplitudes of the present theoretical models form virtually an upper envelope around the observed points, although a few Cepheids show slightly higher amplitudes. Our previously published models lie off the ridge line and possess amplitudes that are generally smaller than the present ones (see Figs. 1 and 3 of Carson and Stothers 1984). It is reassuring that the problem with the amplitudes (if there is a problem) does not affect ϕ_v. Our theoretical calculations show that ϕ_v changes very little during the growth of the pulsations up to limiting amplitude.

By continuing to treat the models as empirical data, we obtain the following mean theoretical relations, through the use of the method of least squares:

$$M_V = \underset{\pm 0.04}{-1.41} - \underset{\pm 0.04}{2.81} \log P , \qquad (14)$$

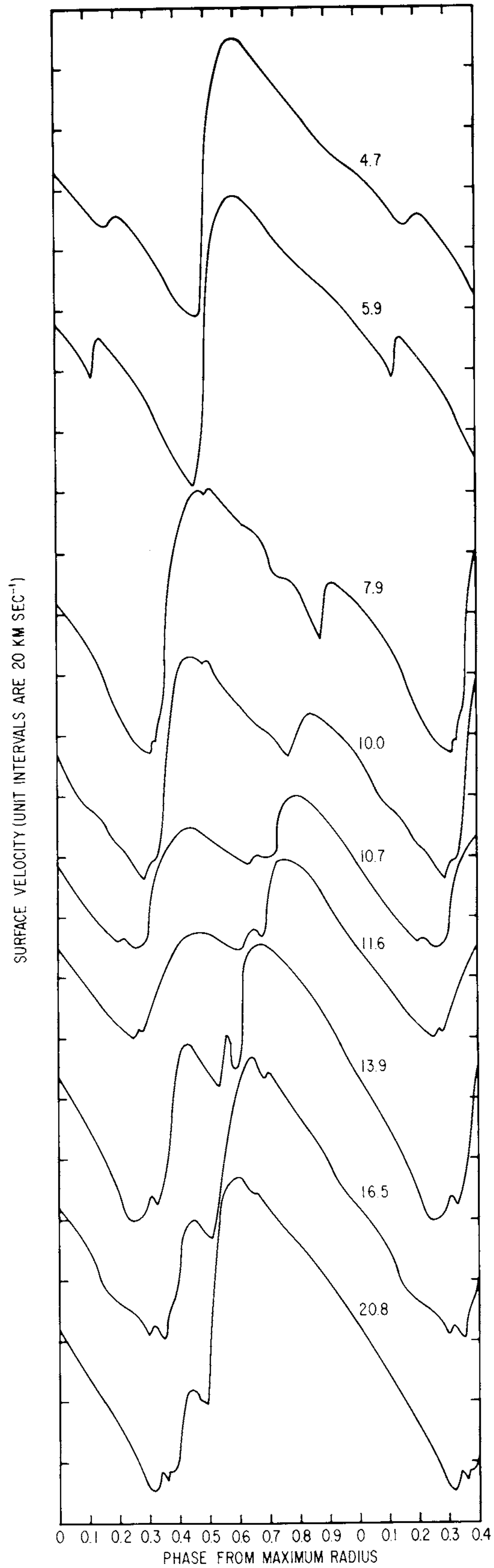

FIG. 1.—Progression of the surface velocity curves with period for theoretical models of ridge-line classical bump Cepheids. Periods are given in days.

$$P\phi_v = (0.211 \pm 0.006)(R/R_\odot)\,, \qquad (15)$$

$$\log (R/R_\odot) = \underset{\pm 0.008}{1.160} + \underset{\pm 0.008}{0.666} \log P\,. \qquad (16)$$

These relations are entirely consistent with the corresponding observational relations, equations (6), (8), and (9).

Effective temperatures of the models can also be compared with observations. Specifically, the two best-studied Cepheids, U Sgr ($P = 6.74$ days) and S Nor ($P = 9.75$ days), show $T_e = 5750$ K and $T_e = 5420$ K, respectively (Pel 1985). Our 6 day model is characterized by $T_e = 5800$ K and our 10 day model has $T_e = 5620$ K. Since the observational mean error is at least 5% (Pel 1985), the models are again consistent with observations. They also satisfactorily follow the ridge line of the theoretical instability strip on the H-R diagram, which lies close to $T_e = 5600$ K for the present luminosities (Cogan 1978; Deupree 1980).

A final comparison with theory that can be made utilizes the masses of Cepheids that have been derived from orbits of Cepheid members of binary systems. Three such masses are now available, although they depend on roughly assumed masses for the companions, owing to the fact that the systems in question do not eclipse and are not visually resolved. Masses for SU Cyg (Evans and Bolton 1987) and for V636 Sco and S Mus (Böhm-Vitense 1986) are plotted in Figure 6 along with our new pulsational relation for bump Cepheids (Table 6). Considering that this theoretical relation is a statistical average for ridge-line Cepheids and that individual Cepheid orbital masses are quite uncertain, agreement is probably as satisfactory as one could expect.

VI. INTERPRETATION OF THE CEPHEID LUMINOSITY EXCESS

All presently testable properties of observed bump Cepheids (with the possible exception of their velocity amplitudes) are consistent with the new models constructed with the Carson opacities. These models, however, require that Cepheids be brighter than standard evolutionary models by $\delta \log (L/L_\odot) = 0.22$, that is, by $\delta M_V = -0.5$ mag.

Observationally, the estimated zero-point error of Cepheid luminosities is almost certainly no larger than ± 0.1 mag, regardless of the size of the uncertainty attached to the Hyades distance modulus (Stothers 1983). Furthermore, the known luminosities of the physical companions to some Cepheids confirm that these binary Cepheids are brighter than standard evolutionary models, although the amount of luminosity excess is not very accurately determined (Böhm-Vitense and Proffitt 1985). Better known is the luminosity excess revealed by other kinds of evolved stars with intermediate to high masses, from their positions in the H-R diagram. Both field stars and cluster and association members show a consistent 0.5 mag excess (Maeder and Mermilliod 1981; Grenier *et al.* 1985; Doom 1985; Stothers and Chin 1985). Therefore, we believe that standard evolutionary tracks are systematically too faint by just this amount.

How can the theoretical evolutionary luminosities be increased? A change of initial chemical composition is one way. To a rough approximation, evolutionary theory predicts the relation:

$$\delta \log (L/L_\odot) \approx 3.5\delta Y - 12\delta Z\,, \qquad (17)$$

based on a variety of published theoretical models of 5 $M_\odot$ stars that are evolving near the blue tip of the core–helium-

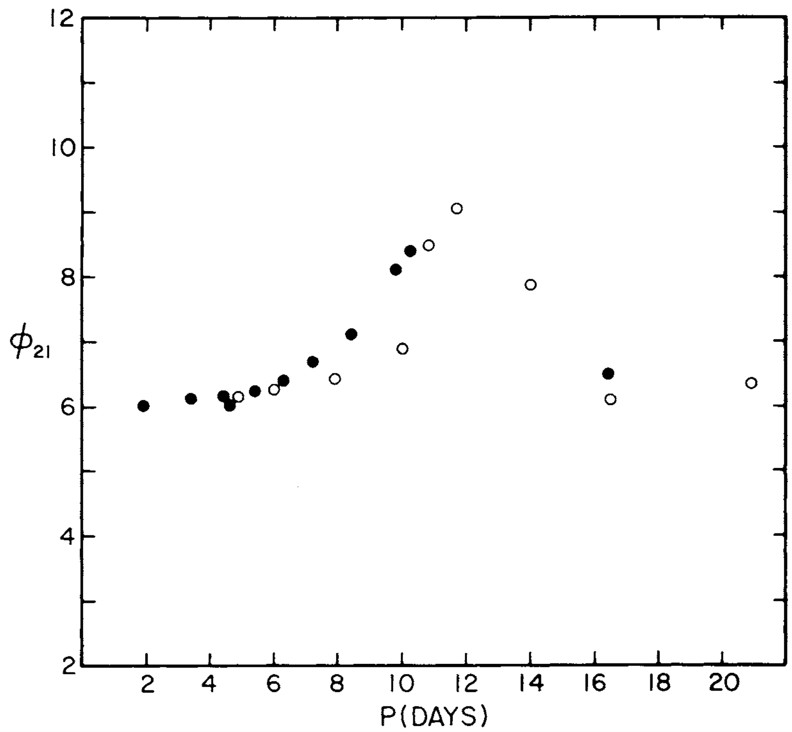

FIG. 2.—Fourier parameter ϕ_{21} vs. period, for theoretical models (*open circles*) and observations (*filled circles*) of classical Cepheids

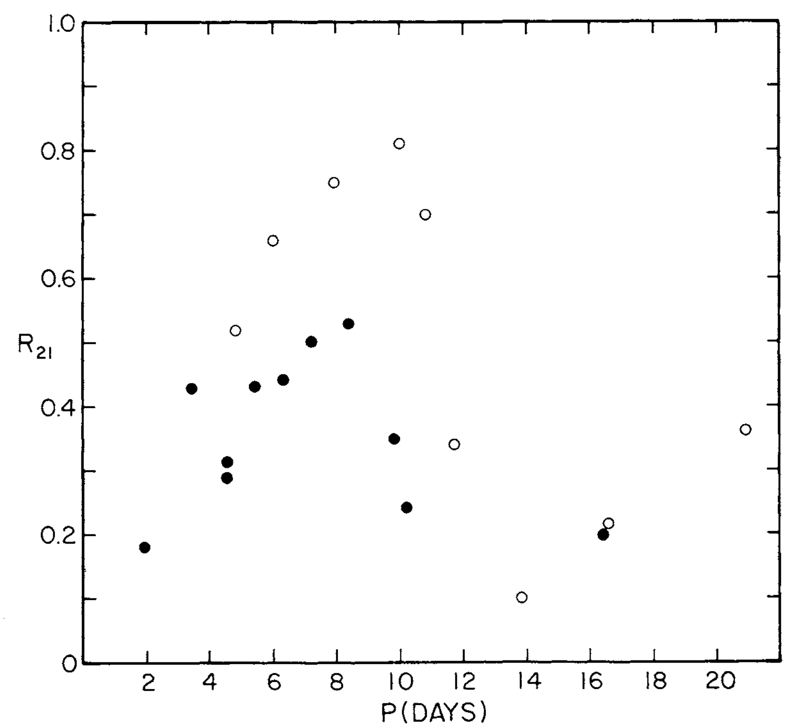

FIG. 3.—Fourier parameter R_{21} vs. period, for theoretical models (*open circles*) and observations (*filled circles*) of classical Cepheids

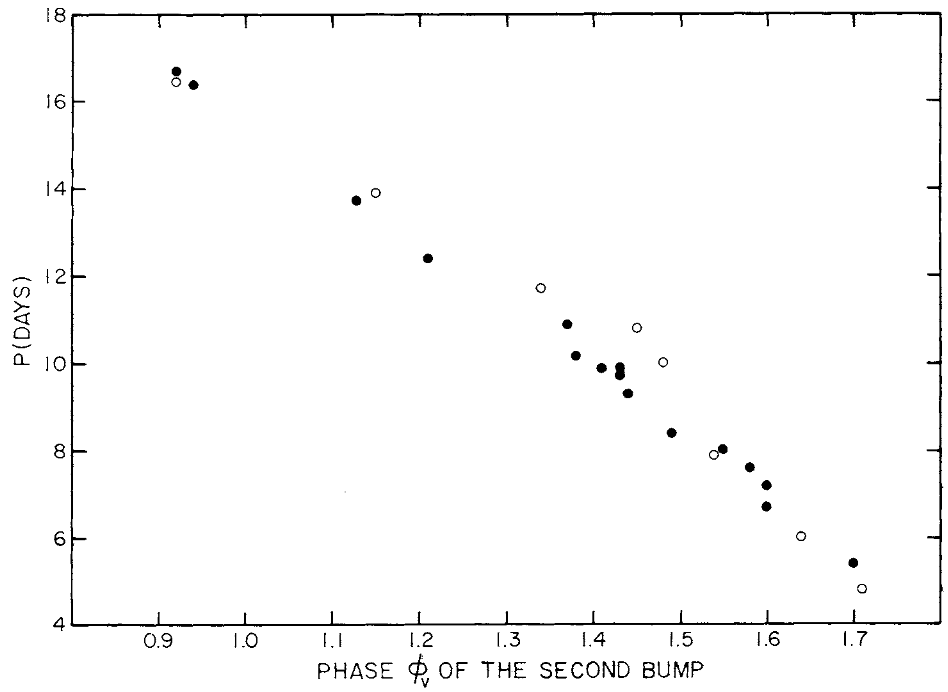

FIG. 4.—Bump phase ϕ_v vs. period, for theoretical models (*open circles*) and observations (*filled circles*) of classical bump Cepheids

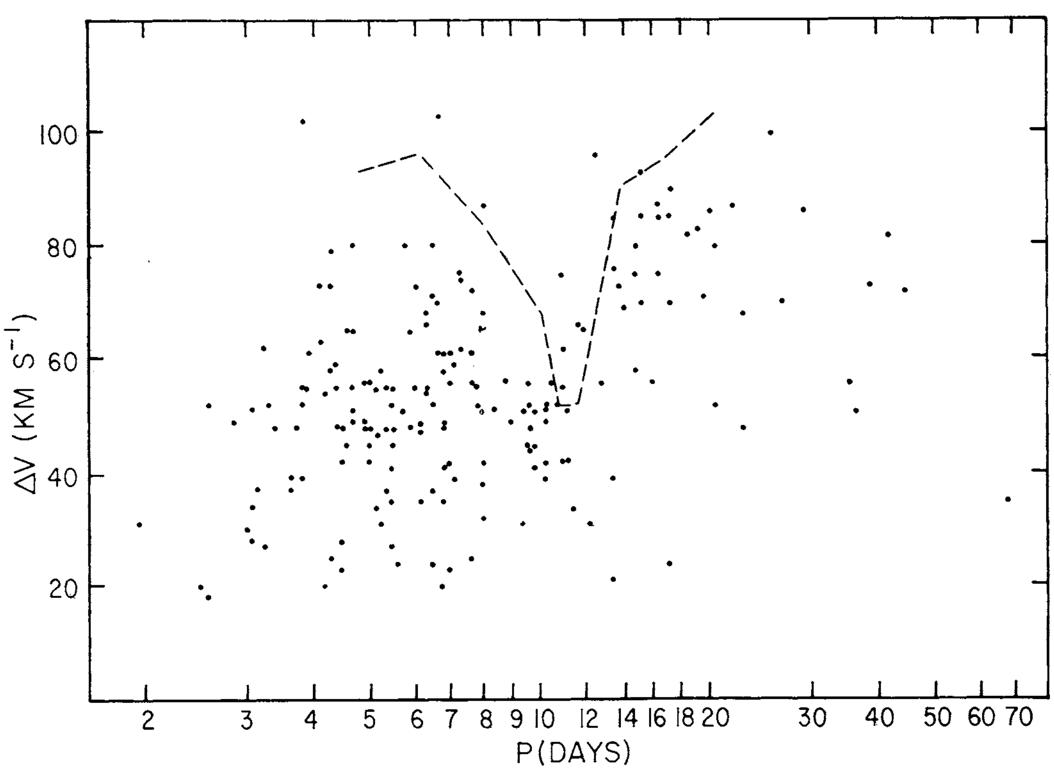

FIG. 5.—Velocity amplitude (radial-velocity amplitude multiplied by 24/17) vs. period for observed classical Cepheids. The dashed line refers to the present models of ridge-line classical bump Cepheids.

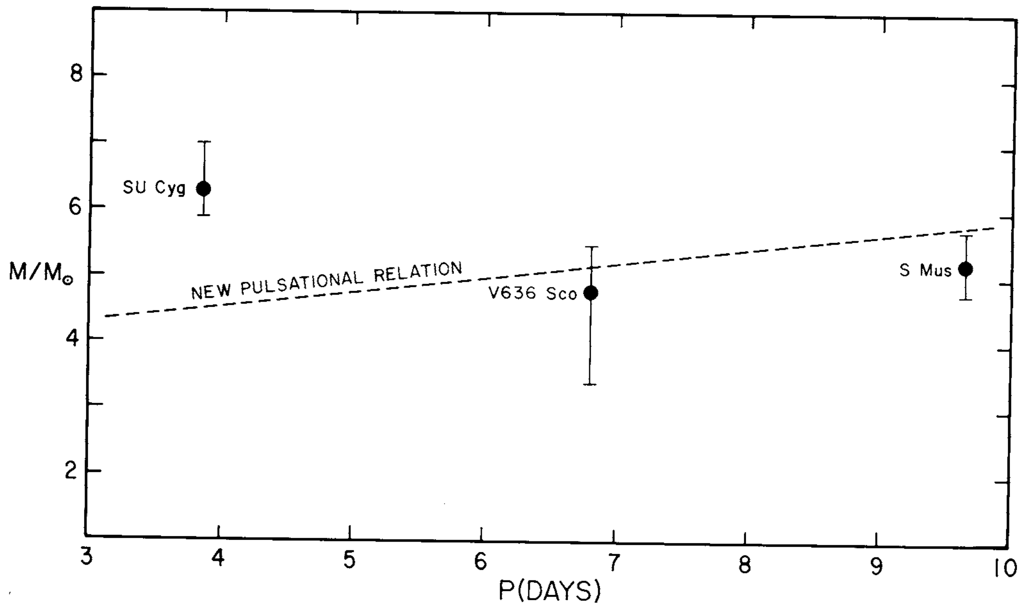

FIG. 6.—Orbital mass vs. pulsation period for three classical Cepheids in spectroscopic binary systems. The dashed line represents the new pulsational masses for ridge-line classical bump Cepheids.

burning loop on the H-R diagram (Hallgren and Cox 1970; Robertson 1971; Becker, Iben, and Tuggle 1977). In a recent review of published stellar and interstellar abundance determinations, Peimbert (1986) has confirmed earlier observational work that Y and Z are correlated, and he has obtained $Y \approx 0.23 + 3.5Z$. Since our standard mass-luminosity relation was based on $Y = 0.24$ and $Z = 0.021$, we find a needed correction to our evolutionary luminosities of $\delta \log (L/L_{\odot}) \approx 0.22$—which is in exact agreement with the correction required by the Cepheid pulsational data. However, "average" values of Y and Z may be statistically more meaningful than a weak (Y, Z) trend line, whose linearity and absolute scale are rather uncertain, especially for large Z. By adopting $\langle Y \rangle = 0.27$ and $\langle Z \rangle = 0.025$ (Stothers 1973; Lacy 1979; Nissen 1980; Wolff and Heasley 1985; Brown 1986; Peimbert 1986), the derived correction to the evolutionary luminosities is only $\delta \log (L/L_{\odot}) \approx 0.06$, which is not enough to meet the whole requirement.

Another possibility is that Cepheids may be evolving in a very late and bright transit of the instability strip. Such late transits, however, are always very fast according to theoretical model calculations, and few Cepheids would be expected to be in such stages. Most Cepheids are probably evolving in the second transit (Becker, Iben, and Tuggle 1977) or the third transit (Cogan 1976), to either of which our adopted luminosities approximately refer.

Convective core overshooting during the main-sequence phase of evolution is the most likely explanation for the excessive brightness. Overshooting increases the mass of the helium-enriched core and hence the luminosity of the star. This type of mixing has already been invoked to explain the general luminosity excess of evolved stars of intermediate and high mass (Maeder and Mermilliod 1981; Grenier *et al.* 1985; Doom 1985; Stothers and Chin 1985) as well as at least part of the traditional bump Cepheid mass discrepancy (Matraka, Wassermann, and Weigert 1982; Becker and Cox 1982; Huang and Weigert 1983; Bertelli, Bressan, and Chiosi 1985). We believe that it probably explains essentially the whole bump Cepheid mass discrepancy.

VII. CONCLUSION

Assuming the validity of the Carson opacities for the main pulsating layers of Cepheids, we have discovered one simple change which can bring the pulsational models into virtually perfect agreement with observed ridge-line classical bump Cepheids. It is necessary only to increase uniformly the theoretical luminosities predicted from standard evolutionary tracks by about 0.5 mag. Interestingly, the need for an increase by exactly this amount has been independently inferred for models of giants and supergiants from their locations in the observational H-R diagram. A 0.5 mag increase is also entirely consistent with modern evolutionary tracks including some degree of convective core overshooting, although it can also be obtained by changing the initial chemical composition. This extent of interagreement implies that the traditionally accepted zero point of observed Cepheid luminosities is correct.

The newly found compatibility between nonlinear pulsational models and observed bump Cepheids may be explicitly seen in comparisons of the progression of individual Cepheid velocity curves with period and in comparisons of the statistical relations between period, luminosity, and radius. Orbital masses of Cepheids also do not disagree, in any significant way, with the new pulsational masses. Thus, the various forms of the Cepheid mass discrepancy (e.g., Cox 1980; Böhm-Vitense 1986) seem to have vanished, at least statistically, except for the persistent problem of the anomalous period ratios in the 2–6 day double-mode (beat) Cepheids, which our new mass-luminosity relation does very little to alleviate (Carson and Stothers 1976). It also remains to be determined whether velocity curves of individual bump Cepheids can be accurately modeled off the ridge line, and especially whether the light curves of any classical Cepheids at all can be reproduced theoretically to the accuracy of the observations.

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